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A primer on the spectral theory of Banach modules and a few of its applications

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Abstract. In this expository note we introduce basic notions of the spectral theory of Banach modules and illustrate how it is used to study different aspects of the theory of localized infinite dimensional linear equations.

Keywords: operator matrices, spectrum, Banach modules, method of similar operators, spectral decay

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Introduction and motivation

The goal of this expository note is to acquaint the readers with basic notions of the spectral theory of Banach modules and some of its important applications. The work is motivated by the study of linear equations of the form

$$Ax = y\ , \tag{1}$$

where x and y belong to Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively, and A is a linear operator from $\mathcal{X}$ to $\mathcal{Y}$. The operator A is typically assumed to be closed but it may or may not be bounded. By $L(\mathcal{X}, \mathcal{Y})$ we denote the Banach space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. As usually, we are interested in the questions of solvability of such equations and the properties of the solutions. The goal is to analyze (i.e. "break into pieces") the input x of the equation (1) its output y, and the operator A, and to understand how "pieces" of the operator A act on certain "pieces" of the input x to produce "relevant pieces" of the output y.

When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, it is natural to pick orthonormal bases $\{x_m\}_{n=1}^N$ and $\{y_m\}_{m=1}^M$ in $\mathcal{X}$ and $\mathcal{Y}$, respectively, and represent the operator A in terms of its matrix $(a_{mn})_{m,n}$, where $a_{mn} = \langle Ax_n, y_m\rangle$, $m = 1, \dots, M$, $n = 1, \dots, N$. As one may recall, we have

$$Ax = \sum_{m=1}^{M} \langle Ax, y_m\rangle y_m = \sum_{m=1}^{M} \left\langle A\left(\sum_{n=1}^{N}\langle x, x_n\rangle x_n\right), y_m\right\rangle y_m = \sum_{m=1}^{M}\sum_{n=1}^{N}\langle x, x_n\rangle a_{mn} y_m.$$

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Thus, the matrix element a_{mn} describes the effect of the "input piece" $\langle x, x_n\rangle x_n$ onto the "output piece" $\langle y, y_m\rangle y_m$.

In a general Banach space, there is no orthonormal basis and one is forced to look for appropriate substitutes. In different instances, one can use Riesz or Schauder bases or various kinds of frames and resolutions of the identity. In all cases, one needs to identify an appropriate substitute for the notion of the support of a vector $x \in \mathcal{X}$ and the notion of restricting each vector to a piece of its support. The spectral theory of Banach modules provides an effective tool of doing just that in a sufficiently general setting. Therefore, we will endow the spaces $\mathcal{X}$ and $\mathcal{Y}$ with additional structures of Banach modules. In this lecture, we will restrict ourselves only to modules over the group algebras $L^1(\mathbb{G})$, where $\mathbb{G}$ is a locally compact Abelian group. We refer to @auxrussian@auxenglish[1] for various properties of such algebras.

Once an analog of a matrix representation is established, one may study how the equation (1) behaves when A, x, and/or y belong to various subclasses. Some of the key questions ask whether a given subclass that A belongs to also contains the inverse operator A^{-1} and the parts of A arising from various factorizations and decompositions. The importance of such questions may be illustrated via the following example.

Example 2.0.1. Let $\mathcal{X} = \mathcal{Y} = \ell^\infty(\mathbb{Z})$ be the Banach space of all bounded sequences indexed by $\mathbb{Z}$. Let also e_n, $n \in \mathbb{Z}$, be the collection of sequences defined by $e_n(m) = \delta elta_{mn}$ – the Kronecker delta. For any $A \in B(\mathcal{X}) = L(\mathcal{X}, \mathcal{X})$ one can define the matrix elements a_{mn} in the usual way: $a_{mn} = (Ae_n)(m)$. Unfortunately, given a bi-infinite matrix, one may not uniquely determine the operator in $B(\mathcal{X})$ that generated that matrix. Indeed, one easily comes up with an example of a non-zero operator that has the zero matrix. Restricting the class of operators allowed in (1) is a way to circumvent this problem.

Banach modules and Beurling spectrum

In this section, we exhibit basic definitions and a few key results of the spectral theory of Banach modules. More details and references to the proofs may be found, for example, in [2].

Definition 2.0.2. Let $\mathfrak{B}$ be a (complex) Banach algebra and $\mathcal{X}$ be a (complex) Banach space. Then $\mathcal{X}$ is a Banach module over $\mathfrak{B}$ if there is a map $(a, x) \mapsto ax : \mathfrak{B} \times \mathcal{X} \to \mathcal{X}$ which has the following properties:

1. $(a + b)\, x = ax + bx$, $a, b \in \mathfrak{B}$, $x \in \mathcal{X}$;
2. $(\alpha a)\, x = \alpha\, (ax) = a\, (\alpha x)$, $\alpha \in \mathbb{C}$, $a \in \mathfrak{B}$, $x \in \mathcal{X}$;
3. $(ab)x = a(bx)$, $a, b \in \mathfrak{B}$, $x \in \mathcal{X}$;
4. $\|ax\| \le \|a\|\|x\|$, $a \in \mathfrak{B}$, $x \in \mathcal{X}$,

As mentioned above, here $\mathfrak{B} = L^1(\mathbb{G})$, i.e. we will only be concerned with L^1-modules. Moreover, we shall impose a few restrictions on the modules we consider.

Assumption 2.0.3. The Banach L^1-module $\mathcal{X}$ is non-degenerate, i.e. given $x \in \mathcal{X} \setminus \{0\}$ there exists $f \in L^1$ such that $fx \neq 0$.

Assumption 2.0.4. The module structure of $\mathcal{X}$ is associated with an isometric representation $\mathcal{T} : \mathbb{G} \to B(\mathcal{X})$. By this we mean

$$\mathcal{T}\,(t)\,(fx) = (T\,(t)\,f)\,x = f(\mathcal{T}(t)x), t \in \mathbb{G}, f \in L^1, x \in \mathcal{X}\ , \tag{2}$$

where T is the translation representation defined by

$$T(t)f(s) = f(t + s),\ \ f \in L^1, t, s \in \mathbb{G}. \tag{3}$$

Recall that a map $\mathcal{T} : \mathbb{G} \to B(\mathcal{X})$ is called a representation if $T(0) = I$ and $T(t+s) = T(t)T(s)$, $t, s \in \mathbb{G}$, and by isometric we mean that $\|T(t)x\| = \|x\|$ for all $t \in \mathbb{G}$ and $x \in \mathcal{X}$.

The Beurling spectrum serves as a perfect proxy for the notion of the support.

Definition 2.0.5. Let $\mathcal{X} = (\mathcal{X}, \mathcal{T})$ be a non-degenerate Banach $L^1(\mathbb{G})$-module, and N be a subset of $\mathcal{X}$. The Beurling spectrum $\Lambda(N) = \Lambda(N, \mathcal{T})$ is defined by

$$\Lambda(N, \mathcal{T}) = \{\lambda \in \widehat{\mathbb{G}} : fx = 0 \text{ for all } x \in N \text{ implies } \widehat{f}(\lambda) = 0, f \in L^1\}\,.$$

Definition 2.0.6. The set $\mathcal{X}_c = \{x \in \mathcal{X} : \text{the map } t \mapsto \mathcal{T}(t)x,\ t \in \mathbb{G}, \text{ is continuous}\}$ is the submodule of $\mathcal{T}$-*continuous* vectors in $\mathcal{X}$. The set $\mathcal{X}_\Phi = L^1\mathcal{X} = \{fx : f \in L^1, x \in \mathcal{X}\}$ is the submodule of *factorizable* vectors in $\mathcal{X}$. We also let $\mathcal{X}_{comp} = \{x \in \mathcal{X} : \Lambda(x) \text{ is compact}\}$.

Proposition 2.0.7. *Assume that $x \in \mathcal{X}_c$. Then,*

$$fx = \int f(t)\,\mathcal{T}(-t)\,x dt,\ \ f \in L^1. \tag{4}$$

Theorem 2.0.8 (Cohen-Hewitt factorization theorem)**.** *One has $\mathcal{X}_c = \mathcal{X}_\Phi = \overline{\mathcal{X}_\Phi} = \overline{\mathcal{X}_{comp}}$.*

Generators of a Banach $L^1(\mathbb{R})$-modules are defined in the following way.

Definition 2.0.9. A closed linear operator $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}$ is called a generator of an $L^1(\mathbb{R})$-module $\mathcal{X}$ if its resolvent $R : \rho(\mathcal{A}) \to B(\mathcal{X})$ satisfies $R(z)x = f_z x$, $z \in \mathbb{C} \setminus \mathbb{R}$, where the Fourier transform of $f_z \in L^1(\mathbb{R})$ is given by $\widehat{f_z}(\lambda) = (\lambda - z)^{-1}$, $\lambda \in \mathbb{R}$.

We remark that under our assumptions all modules have a unique well-defined generator and if $\mathcal{T} : \mathbb{R} \to B(\mathcal{X})$ is a strongly continuous group representation, then $i\mathcal{A}$ is its generator.

Theorem 2.0.10 (Spectral mapping theorem)**.** *One has $\sigma(\mathcal{A}) = \Lambda(\mathcal{X})$ and $\sigma(\mathcal{T}_f) = \overline{\widehat{f}(\Lambda(\mathcal{X}))}$, where $\mathcal{T}_f x = fx$.*

Spectral decay

The Beurling spectrum allows one to introduce various classes of spectral decay for vectors in a Banach module $\mathcal{X} = (\mathcal{X}, \mathcal{T})$. In the following definition, we provide two examples. For simplicity, in this section, we let $\mathbb{G} = \mathbb{R}$. We use a family of functions ϕ_a defined via their Fourier transform by $\widehat{\phi_a}(\xi) = (1 - |\xi - a|)\mathbb{1}_{[-1,1]}(\xi - a)$, $a \in \mathbb{R}$. For more general results we cite [3] and references therein.

Definition 2.0.11. A vector $x \in \mathcal{X}$ has an *exponential spectral decay* if there exist $M > 0$ and $\gamma \in (0, 1)$ such that $\|\phi_n x\| \le M\gamma^n$, $n \in \mathbb{Z}$. A vector $x \in \mathcal{X}$ belongs to the Wiener class $\mathcal{W}(\mathcal{X}, \mathcal{T})$ if $\|x\|_{\mathcal{W}} = \int_{\mathbb{R}} \|\phi_a x\| da < \infty$.

Theorem 2.0.12. *A vector $x \in \mathcal{X}$ has an exponential spectral decay if and only if the function $x_{\mathcal{T}} : \mathbb{R} \to \mathcal{X}$, $x_{\mathcal{T}}(t) = \mathcal{T}(t)x$, $t \in \mathbb{R}$, has a holomorphic extension to a strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < \delta\}$ for some $\delta > 0$. Moreover, $x \in \mathcal{X}_{comp}$ if and only if $x_{\mathcal{T}}$ is a restriction of an entire function of exponential type.*

The study of spectral decay becomes especially useful when $\mathcal{X}$ is a Banach algebra and the representation $\mathcal{T}$ satisfies the following additional condition:

$$\mathcal{T}(t)(xy) = (\mathcal{T}(t)x)(\mathcal{T}(t)y),\ \ t \in \mathbb{R},\ \ x, y \in \mathcal{X}.$$

In the remainder of this section $\mathcal{X}$ is assumed to be such an algebra. As an example, we offer the module $(B(\mathcal{X}_c), \widetilde{\mathcal{T}})$, where

$$\widetilde{\mathcal{T}}(t)A = \mathcal{T}(t)A\mathcal{T}(-t),\ \ t \in \mathbb{R}, A \in B(\mathcal{X}_c) \tag{5}$$

(see [2] for other eligible subclasses of $B(\mathcal{X})$).

Theorem 2.0.13. *Assume that $x \in \mathcal{X}$ has exponential spectral decay and $x^{-1} \in \mathcal{X}$. Then x^{-1} also has exponential spectral decay (albeit with different M and γ).*

Theorem 2.0.14. *The space $\mathcal{W}(\mathcal{X}, \mathcal{T})$ is an inverse closed subalgebra of $\mathcal{X}$.*

The above result is a noncommutative extension of the celebrated Wiener $1/f$ lemma. In the last two decades such extensions received a lot of attention (see e.g. [3–5]) as they allow one to prove localization results for dual frames and prove important sampling theorems [6–9]. Analogs of Theorem 2.0.14 exist for many various classes of spectral decay. The result may also be proved in the setting of operators between different Banach modules, not just for algebras.

Another important topic is an analog of an LU-type factorization for elements in the algebra $\mathcal{X}$. The role of lower and upper triangular matrices is played by the causal and anticausal subalgebras of $\mathcal{X}$ defined by $\mathcal{X}_+ = \{x \in \mathcal{X} : \Lambda(x) = [0, \infty)\}$ and $\mathcal{X}_- = \{x \in \mathcal{X} : \Lambda(x) = (-\infty, 0]\}$, respectively. We refer to [2] for results on causality such as the following theorem.

Theorem 2.0.15. *Assume $x \in \mathcal{X}_c$. Then $x \in \mathcal{X}_+$ if and only if the function $x_\mathcal{T}$ admits a bounded holomorphic extension to the upper halfplane of $\mathbb{C}$.*

Definition 2.0.16. We say that $x \in \mathcal{X}$ admits causal factorization in $\mathcal{X}$ if there exist $x_+, x_- \in \mathcal{X}$ such that $x = x_+x_-$, $x_+, x_+^{-1} \in \mathcal{X}_+$, and $x_-, x_-^{-1} \in \mathcal{X}_-$.

Using Theorem 2.0.15 and factorization results of [10], one may obtain the following result (see [11]).

Theorem 2.0.17. *Assume that $x \in \mathcal{W}(\mathcal{X}, \mathcal{T})$ admits causal factorization in $\mathcal{X}$ and the function $x_\mathcal{T}$ is periodic. Then x admits causal factorization in $\mathcal{W}(\mathcal{X}, \mathcal{T})$.*

The analogs of the above result hold for a large class of subalgebras of $\mathcal{X}$, not just for the Wiener class. Moreover, one can relax the periodicity condition, for example, to almost periodicity. In addition, analogs of Cholesky and QR factorizations also may be considered.

Method of similar operators

The method of similar operators has its origins in various similarity and perturbation techniques. Among them, there are the classical perturbation methods of celestial mechanics, Ljapunov's kinematic similarity method [2, 12**?**], Friedrichs' method of similar operators that is used in quantum mechanics [13], and Turner's method of similar operators [14].

The main idea of the method of similar operators is to construct a similarity transform for the operator $A - B : D(A) \subset \mathcal{X} \to \mathcal{X}$, where A has a relatively simple spectral structure and B is in some sense small compared to A. The goal of the method is to obtain an operator V such that $A - B$ is similar to $A - V$ and the spectral properties of $A - V$ are close to those of A. In particular, certain spectral subspaces of A are mapped by the similarity transform onto certain subspaces that are invariant for $A - V$.

There are plenty of variations of the method of similar operator that are tailored for a specific object of study such as, for example, Dirac operators, Hill operators, operators with an involution, etc. Here we present only one variation, for which we had already introduced most of the necessary terminology and notation. In particular, we assume that A is the generator of the Banach module $(\mathcal{X}, \mathcal{T})$. The operator B is assumed to be bounded. To avoid cumbersome notation, we also assume that the representation $\mathcal{T}$ is strongly continuous.

The main result of an application of the method of similar operators in this case is the following

Theorem 2.0.18. *Under the above conditions, the operator* $A - B$ *is similar to an operator* $A - V$ *with* $V \in (B(\mathcal{X}), \mathcal{T})_{comp}$.

To prove the above result, we use families of functions $\varphi_a, \psi_a \in L^1(\mathbb{R})$, $a > 0$ defined by

$$\varphi_a(t) = \frac{2\sin\frac{3at}{2}\sin\frac{at}{2}}{\pi a t^2},\ t \in \mathbb{R},$$

and

$$\widehat{\psi}_a(\xi) = \frac{1}{\xi}\left(1 - \widehat{\varphi}_a(\xi)\right) = \begin{cases} 0, & |\xi| \le a, \\ -\frac{1}{a} - \frac{1}{\xi}, & -2a < \xi \le -a, \\ \frac{1}{a} - \frac{1}{\xi}, & a < \xi \le 2a, \\ \frac{1}{\xi}, & |\xi| > 2a\ . \end{cases}$$

Since $\|\psi_a\|_1 \to 0$ as $a \to \infty$, there exists $a > 0$ such that $\|\widetilde{\mathcal{T}}_{\psi_a} B\|$ is sufficiently small for the nonlinear map $\Phi : B(\mathcal{X}) \to B(\mathcal{X})$ defined by

$$\Phi(X) = B\widetilde{\mathcal{T}}_{\psi_a} X - (\widetilde{\mathcal{T}}_{\psi_a} X)(\widetilde{\mathcal{T}}_{\varphi_a} X) + B$$

to have a unique fixed point X^* in a ball around B. It may then be verified by direct computation that

$$\big(A - B)(I + \widetilde{\mathcal{T}}_{\psi_a} X^*) = (I + \widetilde{\mathcal{T}}_{\psi_a} X^*)\,(A - V)\ ,$$

where $V = \widetilde{\mathcal{T}}_{\varphi_a} X^* \in (B(\mathcal{X}), \mathcal{T})_{comp}$ because

$$\Lambda(V) \subseteq \ \text{supp}\ \widehat{\varphi}_a = [-2a, 2a]\ .$$

In case when the operator B (i.e. the function $B_{\widetilde{\mathcal{T}}}$) is periodic and sufficiently small, one gets $\Lambda(V) = \{0\}$. As an application of this, one may get conditions that permit a reduction of a non-autonomous (abstract) Cauchy problem to an autonomous one.

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Основные понятия спектральной теории банаховых модулей и некоторые её приложения

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Аннотация. Приводятся базовые понятия спектральной теории банаховых модулей и примеры их использования при изучении локализованных линейных уравнений в бесконечномерных пространствах.

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